p-ADIC AND ADELIC GRAVITY AND COSMOLOGY

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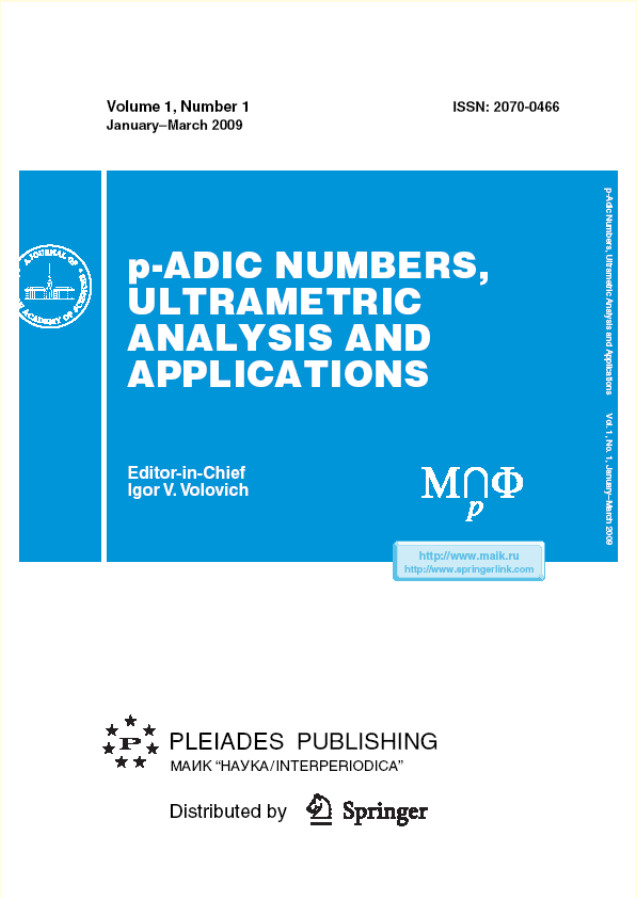
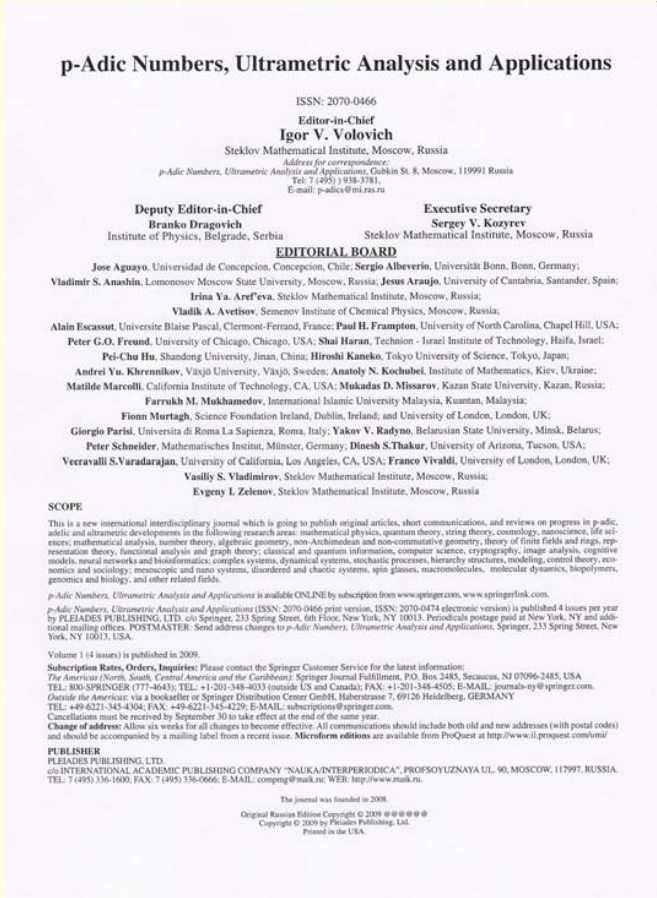
1. Introduction: international interdisciplinary journal "*p*-Adic Numbers, Ultrametric Analysis and Applications"





1. Introduction: international conferences on p-adic mathematical physics

- Workshop on p-adic methods in modeling complex systems, Bielefeld, April 2013.
- 

2nd International Conference on p-Adic Mathematical Physics
Belgrade, 2005

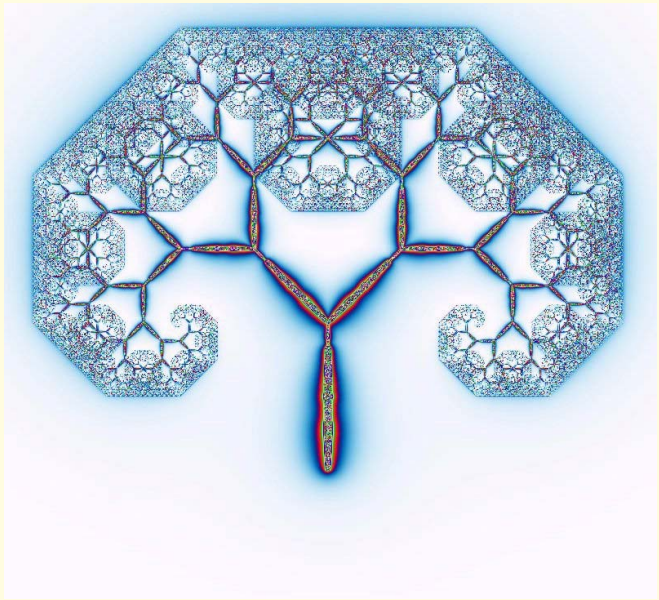
2. p-Adic numbers, adeles and their functions

- discovered by Kurt Hensel (1861-1941) in 1897.
- Any p-adic number ($x \in \mathbb{Q}_p$) has a unique canonical representation

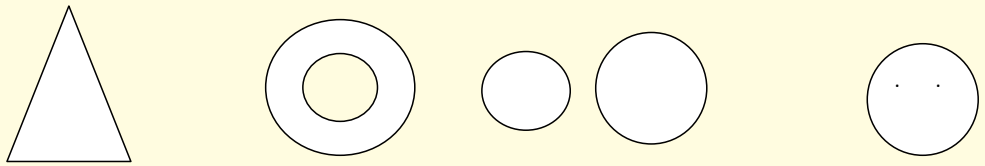
$$x = p^{\nu(x)} \sum_{n=0}^{+\infty} x_n p^n, \ \nu(x) \in \mathbb{Z}, \ x_n \in \{0, 1, \cdots, p-1\}$$

2. p-Adic numbers, adeles and their functions

- p-adic numbers have not natural ordering
- p-adic numbers cannot be completely visualized in real Euclidean space: partial visualization by trees and fractals



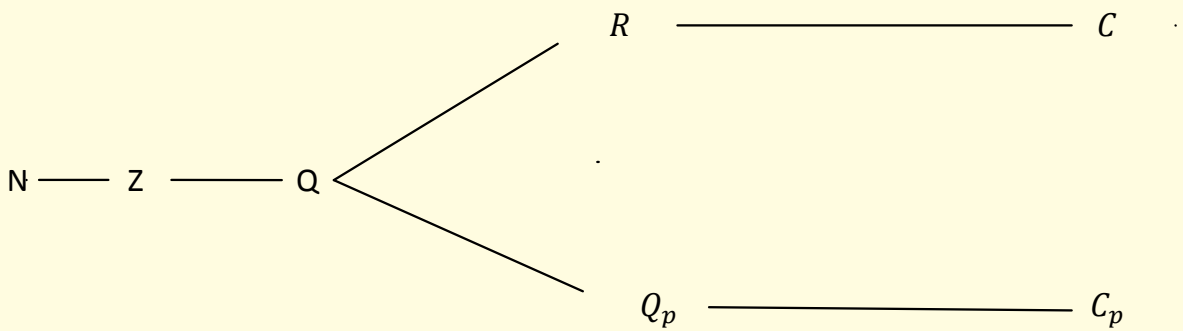
2-ADIC TREE



All triangles are isosceles. *There is no partial intersection of balls.* *Any point of a ball is its center.*

2. p-Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on $\mathbb{Q}$ is equivalent to usual absolute value or to p-adic norm, where p is any prime number ($p = 2, 3, 5, 7, 11, \dots$) .
- p-adic norm of $x \in \mathbb{Q}$: $\quad |x|_p = |p^\nu \frac{a}{b}|_p = p^{-\nu}\,, \quad \nu \in \mathbb{Z}$.
- $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ and $\mathbb{R}$.
- Completion of $\mathbb{Q}$ with respect to p-adic distance gives the field $\mathbb{Q}_p$ of p-adic numbers, in analogous way to construction of the field $\mathbb{R}$ of real numbers.



2. p-Adic numbers, adeles and their functions

- There are mainly two kinds of analysis of p-adic variable:
  (*i*) *p-adic valued functions of p-adic variable*
  (*ii*) *complex (real) valued functions of p-adic variable.*
- *Usual complex-valued functions of p-adic argument are:*

$$\pi_s(x) = |x|_p^s, \quad \chi_p(x) = \exp 2\pi i\{x\}_p, \quad \Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1 \\ 0, & |x|_p > 1. \end{cases}$$

- *Analysis of complex (real) valued functions of p-adic (and real) variables is necessary to connect models with measurements.*

2. p-Adic numbers, adeles and their functions

- Real and p-adic numbers are unified by adeles. An adele α is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \ \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set $\mathcal{P}$ of primes p one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. p-adic integers.

- Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p .$$

Connection of p-adic and real properties of rational numbers

$$|x|_\infty \times \prod_{p\in\mathbb{P}} |x|_p = 1\,, \text{ if } x \in \mathbb{Q}^\times$$

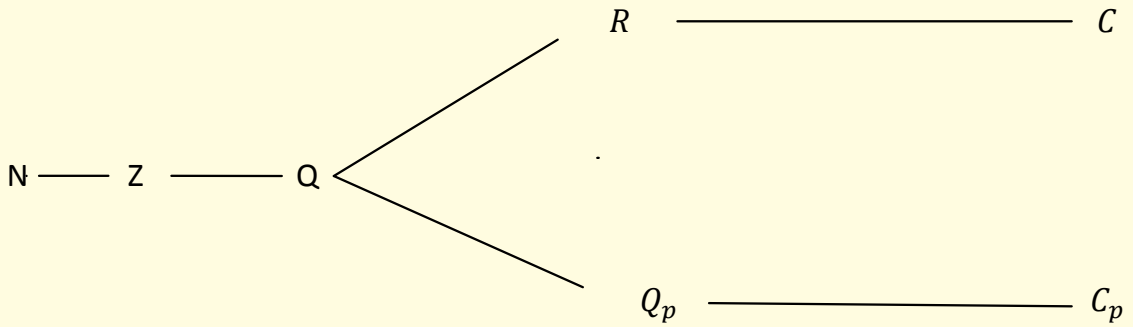
$$\chi_\infty(x) \times \prod_{p\in\mathbb{P}} \chi_p(x) = 1\,, \text{ if } x \in \mathbb{Q}$$

$$\chi_\infty(x) = \exp(-2\pi i x), \quad \chi_p(x) = \exp 2\pi i\{x\}_p$$

2. Rational numbers and measurements

- Any measurement can be viewed as measurement of a distance.
- Measurement of a distance is comparison of its length with unit length.
- Result of measurement is a rational number with an error.
- This rational number is a real, but not a *p*-adic, number.
- Measurement is related to the Archimedes axiom in geometry.
- According to quantum gravity one cannot measure distances smaller than the the Planck length: $\Delta x \geq \ell_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33} cm$. This is limit for application of real numbers in micro-world and an window for application of *p*-adic numbers and adeles.

2. p-Adic numbers, adeles and their applications



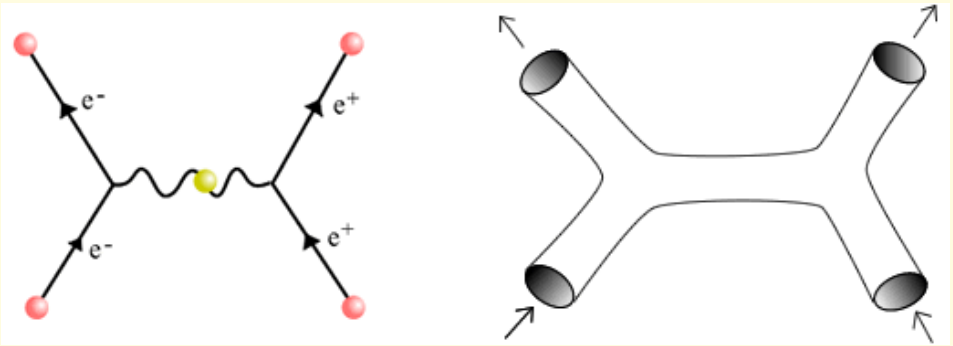
Applications must contain complex (real) valued functions of p-adic (or adelic) argument. At a hidden level applications can also contain p-adic (or adelic) valued functions of p-adic (or adelic) valued arguments.

- At the Planck scale and string theory.
- Gravity theory and cosmology.
- In bioinformation systems – genetic code, ...
- In very complex systems with hierarchical structure.
- Some other applications.

3. p-Adic and adelic string theory

Volovich, Vladimirov, Freund, Witten, Dragovich, ...

Electron (e^-) and positron (e^+) scattering *in quantum field theory* and *string theory*.



3. p-Adic and adelic string theory

String amplitudes:

- standard crossing symmetric Veneziano amplitude

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x$$
$$= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}$$

- p-adic crossing symmetric Veneziano amplitude

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x$$
$$= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

3. p-Adic and adelic string theory

- Freund-Witten product formula for adelic strings

$$A(a,b) = A_\infty(a,b)\prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const.$$

- Amplitude for real string $A_\infty(a,b)$, which is a special function, can be presented as product of inverse p-adic amplitudes, which are elementary functions.

3. p-Adic and adelic string theory

- There is an effective field description of scalar open and closed p-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher (Koba-Nielsen) ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric with signature $(-\,+\,...\,+)$ (Freund, Witten, Frampton, Okada, ...) .

3. p-Adic and adelic Einstein gravity

- Einstein Theory of Gravity (ETG) and Quantum Theory lie in foundation of modern theoretical physics.
- General theory of relativity is Einstein theory of gravity..
- At the cosmic scale there is only gravitational force (interaction). Hence gravitational force governs dynamics of the Universe as a whole.
- Cosmology is a science about the Universe as a whole.
- There are cosmic observational results which have not generally accepted explanation. Two of these observations are: accelerated expansion of the Universe and large velocities of stars in spiral galaxies.

3. p-Adic and adelic Einstein gravity

- In ETG gravity is presented in terms of the pseudo-Riemannian geometry.
- In (pseudo-)Riemannian geometry distance is defined by $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ with respect to some system of coordinates.
- From metric tensor ($g_{\mu\nu}$) one can obtain any information about Riemannian space.

$$g_{\mu\nu} \to \Gamma^{\alpha}_{\mu\nu} \to R^{\alpha}_{\beta\mu\nu} \to R_{\mu\nu} \to R$$

- Christoffel symbol

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu\, g_{\beta\nu} + \partial_\nu\, g_{\mu\beta} - \partial_\beta\, g_{\mu\nu})$$

3. p-Adic and adelic Einstein gravity

- Riemann-Christoffel curvature tensor

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\beta}$$

- Ricci tensor and Ricci scalar

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}\,, \qquad R = g^{\mu\nu}\,R_{\mu\nu}$$

- Einstein equations for gravity field (metric tensor)

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu} - \Lambda\,g_{\mu\nu}$$

- Einstein-Hilbert action (g is determinant of $g_{\mu\nu}$)

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\Lambda)d^4x\,, \qquad (c = 1)$$

3. p-Adic and adelic Einstein gravity

- One can formally consider main ingredients of (pseudo)Riemannian geometry and Einstein theory of gravity as p-adic valued. It has not direct physical meaning, but can be used in argument of the path integrals.
- We now consider Quantum Cosmology (QC). In QC we are interested in wave function of the Universe, which contains quantum information about the Universe at its very early stage of evolution.
- We use adelic quantum mechanics to obtain adelic wave function of the Universe.
- As a simple example we consider the de Sitter model of the Universe.

3. p-Adic and adelic Einstein gravity

- Adelic space-time and adelic (pseudo)Riemannian geometry

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \ \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set $\mathcal{P}$ of primes p one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. p-adic integers.

- Adele α can be space-time point or any ingredient of (pseudo)Riemannian geometry, including gravity action.
- Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p .$$

4. p-Adic and adelic quantum cosmology

- Suppose that at the Planck scale ($10^{-33} cm$) there are real and p-adic strings, and that the space-time is adelic. Then it means that the Universe at the very beginning was adelic and it gives rise to consider adelic quantum cosmology.
- Adelic quantum cosmology is application of adelic quantum mechanics to the cosmological models.
- Adelic quantum mechanics can be viewed as a triple $(L_2(\mathbb{A}), W, U(t))$, where $L_2(\mathbb{A})$ is the Hilbert space on $\mathbb{A}$, W means Weyl quantization, and $U(t)$ is unitary evolution operator on $L_2(\mathbb{A})$.
- $U(t)$ can be expressed trough its kernel $\mathcal{K}(x'', t''; x', t')$

$$\Psi_{\mathcal{P}}(x'', t'') = \int \mathcal{K}(x'', t''; x', t')\Psi_{\mathcal{P}}(x', t')dx',$$

where

$$\Psi_{\mathcal{P}}(x'', t'') = \psi_{\infty}(x_{\infty}, t_{\infty}) \prod_{p \in \mathcal{P}} \psi_p(x_p, t_p) \prod_{p \notin \mathcal{P}} \Omega(|x_p|_p))$$

is adelic eigenfunction, and

$$\mathcal{K}(x'', t''; x', t') = \mathcal{K}_{\infty}(x''_{\infty}, t''_{\infty}; x'_{\infty}, t'_{\infty}) \prod_{p} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p)$$
$$= \prod_{v} \mathcal{K}_v(x''_v, t''_v; x'_v, t'_v)$$

is adelic transition probability amplitude.

4. p-Adic and adelic quantum cosmology

$\mathcal{K}(x'', t''; x', t')$ is naturally realized by Feynman's path integral

$$\mathcal{K}(x'', t''; x', t') = \int \chi_{\mathbb{A}} \left(-\frac{1}{h} S_{\mathbb{A}}[q] \right) \mathcal{D}_{\mathbb{A}} q$$

$$= \prod_v \int \chi_v \left(-\frac{1}{h} \int_{t'_v}^{t''_v} L(\dot{q}_v, q_v)\, dt_v \right) \mathcal{D} q_v$$

Vacuum state $\Omega(|x''_p|_p)$ has property

$$\Omega(|x''_p|_p) = \int_{\mathbb{Q}_p} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p) \Omega(|x'_p|_p) dx'_p = \int_{\mathbb{Z}_p} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p) dx'_p$$

4. p-Adic and adelic quantum cosmology

Some interesting results of adelic quantum modeling:

- Connection between adelic harmonic oscillator and Riemann zeta function
- Discreteness of space and time at the Planck scale.

4. p-Adic and adelic quantum cosmology

- The Einstein-Hilbert action for the de Sitter model of the Universe is

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}K$$

- Large scale space is homogenous and isotropic,and described by the FLRW metric

$$ds^2 = \sigma^2\Big[-N^2dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big)\Big]$$

$$ds^2 = \sigma^2[-\frac{N^2}{q(t)}dt^2 + q(t)d\Omega_3^2], \quad \sigma^2 = \frac{2G}{3\pi}$$

4. p-Adic and adelic quantum cosmology

- The corresponding action is

$$S_v[q] = \frac{1}{2}\int_{t'}^{t''} dtN\Big(-\frac{\dot{q}^2}{4N^2} - \lambda q + 1\Big), \quad \lambda = \frac{2\Lambda G}{9\pi}$$

- For $N = 1$, the equation of motion $\ddot{q} = 2\lambda$ has solution

$$q(t) = \lambda t^2 + \Big(\frac{q'' - q'}{T} - \lambda T\Big)t + q',$$

which presents classical trajectory, where $q'' = q(t'')$, $q' = q(t')$, $T = t'' - t'$. This solution resembles the motion of a particle in a constant field.

4. p-Adic and adelic quantum cosmology

- The corresponding action for classical trajectory (path) is

$$\bar{S}_v(q'', T; q', 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q'+q'')-2]\frac{T}{4} - \frac{(q''-q')^2}{8T}, \quad v = \infty, 2, 3, .$$

- The corresponding transition amplitude is

$$\mathcal{K}_v(q'', T; q', 0) = \frac{\lambda_v(-8T)}{|4T|_v^{\frac{1}{2}}} \chi_v(-\bar{S}_v(q'', T; q', 0))$$

- The corresponding vacuum state $\Omega(|q|_p)$ follows from

$$\int_{|q'|_p \leq 1} \mathcal{K}_p(q'', T; q', 0)\, dq' = \Omega(|q''|_p)$$

4. p-Adic and adelic quantum cosmology

- The p-adic Hartle-Hawking wave function is given by

$$\Psi_p(|q|_p) = \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{\frac{1}{2}}} \chi_p\Big(-\frac{\lambda^2 T^3}{24}[\lambda q - 2]\frac{T}{4} + \frac{q^2}{8T}\Big)$$

  and gives vacuum state $\Omega(|q|_p)$ with the condition $\lambda = 3 \cdot 4 \cdot \ell, \quad \ell \in \mathbb{Z}$.

- Existence of the vacuum state for all (or almost all primes p is a necessary condition for a model to be adelic).
- Any adelic quantum model provides some discreteness. In adelic quantum cosmology length of discreteness is the Planck length.

4. p-Adic and adelic quantum cosmology

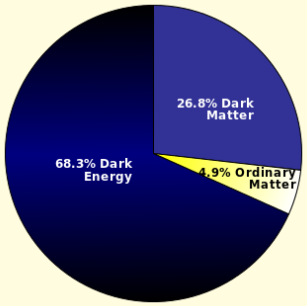
- Discreteness at the Planck scale

$$\Psi(q) = \Psi_\infty(q) \prod_p \Omega(|q|_p) = \begin{cases} \Psi_\infty(q), & q \in \mathbb{Z} \\ 0, & q \in \mathbb{Q} \setminus \mathbb{Z}. \end{cases}$$

It means $q = n \cdot \ell_P$, where $\ell_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} m$.

4. *p*-Adic and adelic origin of dark matter and dark energy

- If Einstein general theory of relativity is theory of gravity for the entire Universe then there is only about 5% of ordinary matter and 95% of matter in the Universe of unknown nature, i.e 95% of the Universe is dark.
- *Dark matter* (27%) should be responsible for rotation velocities of galaxies.
- *Dark energy* (68%) was introduced in 1998 as a possibility to explain Universe expansion acceleration.
- It is possible that there is some *p*-adic matter in the Universe, and that dark matter and dark energy have *p*-adic origin, i.e. origin in *p*-adic strings.



3. p-Adic and adelic quantum cosmology

- In the last decade there are a lot of papers related to modification of the Einstein theory of gravity to find alternative (without dark energy) explanation of accelerated expansion of the Universe.
- Nonlocal modified gravity is one of attractive approaches to generalization of Einstein theory of gravity, and its motivation is also in p-adic string theory.

5. Concluding remarks: Main references

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5. Concluding remarks: Main references

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5. Concluding remarks

- Numerical results of experiments are real rational numbers – they are not p-adic numbers.
- There are some interesting and promising applications of p-adic analysis in p-adic mathematical physics (string theory, gravity and cosmology, ...) and biology (genetic code,...).
- We conjecture that p-adic numbers will play significant role in some complex systems, like complex numbers play role in quantum theory.